Discrete-Time k-Positive Linear Systems

Rola Alseidi, Michael Margaliot, and Jürgen Garloff

***Abstract*—Positive systems play an important role in systems and control theory and have found many applications in multiagent systems, neural networks, systems biology, and more. Positive systems map the nonnegative orthant to itself (and also the nonpositive orthant to itself). In other words, they map the set of vectors with zero sign variations to itself. In this article, discrete-time linear systems that map the set of vectors with up to $k-1$ sign variations to itself are introduced. For the special case $k=1$ these reduce to discrete-time positive linear systems. Properties of these systems are analyzed using tools from the theory of sign-regular matrices. In particular, it is shown that almost every solution of such systems converges to the set of vectors with up to $k-1$ sign variations. It is also shown that these systems induce a positive dynamics of k-dimensional parallelotopes.**

***Index Terms*—Compound matrices, cones of rank k, exterior products, sign-regular matrices, stability analysis.**

I. Introduction

THE discrete-time (DT) linear time-varying (LTV) system

$$x(i+1)=A(i)x(i), \quad x(0)=x_0 \in \mathbb{R}^n \tag{1}$$

is called *positive* if and only if (iff) it maps the nonnegative orthant

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in \{1,\dots,n\}\}$$

to itself. This holds iff $A(i) \geq 0$ (i.e., all the entries of $A(i)$ are nonnegative) for all $i \geq 0$. Note that a positive system also maps the nonpositive orthant $\mathbb{R}^n_- := -\mathbb{R}^n_+$ to itself. In other words, it maps the set of vectors with zero sign variations to itself. The system (1) is called *strongly positive* if it maps $\mathbb{R}^n_+ \setminus \{0\}$ to $\text{int}(\mathbb{R}^n_+)$ (the interior of $\mathbb{R}^n_+$).

Positive systems appear naturally when the state-variables represent quantities that can only take nonnegative values, e.g., probabilities, concentrations of molecules, densities of particles, etc. Positive LTVs play an important role in linear systems and control theory (see, e.g., [9] and [31]) and via *differential analysis* [13], [24], also in the analysis of nonlinear systems. To explain this, consider the *nonlinear* time-varying system

$$x(i+1)=f(i,x(i)) \tag{2}$$

and suppose that its trajectories evolve on a convex state-space $\Omega \subseteq \mathbb{R}^n$, and that f is C^1 with respect to x. For $y \in \Omega$, let $x(i,y)$ denote the solution of (2) at time i for $x(0)=y$. Pick $a,b \in \Omega$, and let

$$z(i) := x(i,a)-x(i,b)$$

that is, the difference at time i between the trajectories emanating from a and from b at time zero. Then

$$z(i+1) = f(i,x(i,a)) - f(i,x(i,b)) = J^{ab}(i)z(i)$$

$$\text{with } J^{ab}(i) := \int_0^1 \frac{\partial}{\partial x} f(i, rx(i,a)+(1-r)x(i,b))\,\mathrm{d}r. \tag{3}$$

If $J^{ab}(i) \geq 0$ for all $a,b \in \Omega$, and all $i \geq 0$, then the *variational system* (3) is a positive LTV, and this has important consequences for the behavior of (2). Roughly speaking, almost every bounded trajectory of a smooth strongly positive system converges to a periodic trajectory (a cycle) [30]. This is quite different from the behavior in the continuous-time (CT) case, where almost every bounded trajectory of the nonlinear system converges to the set of equilibria [36].

The dynamics of a DT positive LTV maps the set of vectors with zero sign variations to itself. A natural question is: what systems map the set of vectors with up to $k-1$ sign variations to itself? We call such a system a *DT k-positive system*. Then, a 1-positive system is just a positive system, but for $k>1$ the system may be k-positive yet not positive.

CT k-positive systems have been recently defined and analyzed in [38]. In the CT and time-invariant case, i.e., $\dot{x}(t)=Ax(t)$, the *matrix exponential* of A should satisfy for all time a property called strict sign-regularity of order k, for the definition see the next paragraph, and this can be tested easily by checking sign conditions on the entries of A itself [38]. In the DT case studied here, the matrix A itself must have this property, and verifying this is nontrivial.

A matrix $A \in \mathbb{R}^{n\times m}$ is called *sign-regular of order k* (denoted by SR_k) if all its minors of order k, i.e., determinants of its $k \times k$ submatrices, are nonnegative or all are nonpositive. For example, if all the entries of A are nonnegative then it is SR_1. A matrix is called *strictly sign-regular of order k* (denoted by SSR_k) if it is SR_k, and all the minors of order k are nonzero. In other words, all minors of order k are nonzero and have the same sign.[1] To refer to the common sign of the minors of order k, we introduce the *signature* $\epsilon_k \in \{-1,1\}$.

[1]We note that the terminology in this field is not uniform and some authors refer to such matrices as *sign-consistent of order k*.

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For example, the matrix

$$A := \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0.1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

is SR_1 (but not SSR_1 as some entries are zero). It has both positive and negative 2-minors (e.g., $\det(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}) = 1$ $\det(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix})] = -2$), so it is not SR_2. All its 3-minors are positive, so it is SSR_3 with signature $\epsilon_3 = 1$, and $\det(A) > 0$, so it is SSR_4 with $\epsilon_4 = 1$.

After the first consideration of SR_k matrices in [20], these matrices have been the subject of only a few studies. In [1], the authors analyze the spectral properties of nonsingular matrices that are SSR_k for a specific value of k. These results are extended to matrices that are SSR_k for several values of k, for example for all odd k.

A matrix $A \in \mathbb{R}^{n\times m}$ is called *[strictly] sign-regular* ([S]SR) if it is $[\mathrm{S}]\mathrm{SR}_k$ for all $k = 1, \dots, \min\{n, m\}$ [28] and [29, p. 86]. The most important examples of SR [SSR] matrices are the totally nonnegative (TN) [totally positive (TP)] matrices, that is, matrices with all minors nonnegative [positive]. Such matrices have applications in numerous fields including approximation theory, combinatorics, probability theory, computer aided geometric design, differential and integral equations, and more [8], [16], [20] and [29].

A very important property of SSR matrices is that multiplying a vector x by such a matrix cannot increase the number of sign variations in x [16]. To explain this *variation diminishing property* (VDP), we introduce some notation. For $y \in \mathbb{R}^n \setminus \{0\}$, let $s^-(y)$ denote the number of sign variations in y after deleting all its zero entries with $s^-(0)$ defined as zero. For $y \in \mathbb{R}^n$, let $s^+(y)$ denote the maximal possible number of sign variations in y after each zero entry is replaced by either $+1$ or -1. For example, for $n = 4$ and $y = [1\ -1\ 0\ -\pi]^T$ (where the superscript T denotes transposition), we have $s^-(y) = 1$ and $s^+(y) = 3$. Obviously,

$$0 \le s^-(y) \le s^+(y) \le n - 1 \text{ for all } y \in \mathbb{R}^n. \tag{4}$$

The first important results on the VDP of matrices were obtained by Fekete and Pólya [10] and Schoenberg [34]. Later on, Gantmacher and Krein [16, Ch. V] elaborated rather completely the various forms of VDPs and worked out the spectral properties of SR matrices. Two important examples of such VDPs are: if $A \in \mathbb{R}^{n\times m}$ $(m \le n)$ is SR and of rank m then

$$s^-(Ax) \le s^-(x) \text{ for all } x \in \mathbb{R}^m$$

whereas if A is SSR then

$$s^+(Ax) \le s^-(x) \text{ for all } x \in \mathbb{R}^m \setminus \{0\}.$$

Thus, if A is SSR then both $s^-(x(i))$ and $s^+(x(i))$ are integer-valued functions that do not increase along solutions of $x(i+1) = Ax(i)$.

For $k \in \{1, \dots, n\}$, let

$$\begin{aligned} P_-^k &:= \{z \in \mathbb{R}^n : s^-(z) \le k-1\} \\ P_+^k &:= \{z \in \mathbb{R}^n : s^+(z) \le k-1\}. \end{aligned} \tag{5}$$

Then, positive systems map the set P_-^1 to P_-^1, whereas strongly positive systems map $P_-^1 \setminus \{0\}$ to P_+^1. This naturally leads to the question: which linear systems map P_-^k to P_-^k and which map $P_-^k \setminus \{0\}$ to P_+^k? In this article, we define and analyze such systems, called DT k-positive linear systems. We show that such systems have interesting dynamical properties that generalize the properties of positive systems.

The remainder of this article is organized as follows. In Section II, we review notations, definitions, and basic properties that will be used later on. Section III defines DT k-positive linear systems and analyzes their properties. Finally, this article concludes in Section IV. In passing, we note that our results are part of a growing body of research on the applications of sign-regularity (and, in particular, total positivity) to dynamical systems [1], [4], [22], [25], [26], [35], and [38].

II. Preliminaries

This section is divided into three subsections. Sections II-A and II-B introduce definitions and notations needed later on. Section II-C reviews the structure of the invariant sets P_+^k and P_-^k.

A. Basic Notation and Definitions

For an integer $n \ge 1$ and $k \in \{1, \dots, n\}$, let $Q_{k,n}$ denote the set of all strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$. For example, $Q_{2,3} = \{12, 13, 23\}$.

For $A \in \mathbb{R}^{n\times m}$, $\alpha \in Q_{k,n}$, and $\beta \in Q_{j,m}$, we denote the submatrix of A lying in the rows indexed by α and columns indexed by β by $A[\alpha, \beta]$. Thus, $A[\alpha, \beta] \in \mathbb{R}^{k\times j}$. If $k = j$, then, we set

$$A(\alpha|\beta) := \det(A[\alpha, \beta])$$

that is, the minor corresponding to the rows indexed by α and columns indexed by β. We often suppress the brackets associated with an index sequence if we enumerate its entries explicitly.

B. Multiplicative Compound

Let $A \in \mathbb{R}^{n\times m}$. For any $k \in \{1, \dots, \min\{n, m\}\}$, the *kth multiplicative compound* of A is the $\binom{n}{k} \times \binom{m}{k}$ matrix that includes all the minors of order k of A organized in lexicographic order. For example, if $A \in \mathbb{R}^{3\times 3}$ then

$$A^{(2)} = \begin{bmatrix} A(12|12) & A(12|13) & A(12|23) \\ A(13|12) & A(13|13) & A(13|23) \\ A(23|12) & A(23|13) & A(23|23) \end{bmatrix}.$$

Note that $A^{(1)} = A$ and that if $m = n$, then $A^{(n)} = \det(A)$. Note also that A is SSR_k $[\mathrm{SR}_k]$ if either $A^{(k)} > 0$ or $A^{(k)} < 0$ [either $A^{(k)} \ge 0$ or $A^{(k)} \le 0$].

The Cauchy–Binet formula [8, Th. 1.1.1] provides an expression for the minors of the product of two matrices. Pick $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times m}$. Let $C := AB$. Pick $k \in \{1, \dots, \min\{n, p, m\}\}$, $\alpha \in Q_{k,n}$, and $\beta \in Q_{k,m}$. Then

$$C(\alpha|\beta) = \sum_{\gamma \in Q_{k,p}} A(\alpha|\gamma) B(\gamma|\beta). \tag{6}$$

For $n = p = m$ and $k = n$ this reduces to the familiar formula $\det(AB) = \det(A)\det(B)$. Note that (6) implies that

$$(AB)^{(k)} = A^{(k)} B^{(k)} \tag{7}$$

for all $k \in \{1, \dots, \min\{n, p, m\}\}$. This justifies the term multiplicative compound.

C. Sets of Vectors With Sign Variations

Consider the sets defined in (5). It is well-known (see, e.g., [29, Ch. 3]) that if $v^i \in \mathbb{R}^n$, $i = 1, 2, \dots$, is a set of vectors such that $v := \lim_{i\to\infty} v^i$ exists then

$$s^-(v) \le \liminf_{i\to\infty} s^-(v^i) \le \limsup_{i\to\infty} s^+(v^i) \le s^+(v). \tag{8}$$

Intuitively speaking, this is because we only need to consider what happens when the limit vector v includes zero entries, and such entries can only decrease s^- and can only increase s^+.

The next useful result is well-known, but for the sake of completeness we include its proof in the Appendix.

Fact 1: The set P_-^k is closed and

$$P_+^k = \text{int}(P_-^k). \tag{9}$$

It is clear that

$$P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n, \quad P_+^1 = \text{int}(\mathbb{R}_+^n) \cup \text{int}(\mathbb{R}_-^n). \tag{10}$$

Also, the sets are nested, as

$$P_-^1 \subset P_-^2 \subset \cdots \subset P_-^n = \mathbb{R}^n$$
$$P_+^1 \subset P_+^2 \subset \cdots \subset P_+^n = \mathbb{R}^n. \tag{11}$$

If $x \in P_-^k$ then $rx \in P_-^k$ for all $r \in \mathbb{R}$, and if $x \in P_+^k$ then $sx \in P_+^k$ for all $s \in \mathbb{R} \setminus \{0\}$, so both P_-^k and $P_+^k \cup \{0\}$ are cones. Yet, in general P_-^k and P_+^k are *not* convex sets. For example, for $n = 2$ and the vectors $x := [2\ 0]^T$, $y := [0\ {-2}]^T$, we have $x, y \in P_-^1$ yet $\frac{x}{2} + \frac{y}{2} = [1\ {-1}]^T \notin P_-^1$.

Recall that a set $C \subseteq \mathbb{R}^n$ is called *a cone of rank k* [23] if

(i) C is closed.
(ii) $x \in C$ implies that $rx \in C$ for all $r \in \mathbb{R}$.
(iii) C contains a linear subspace of dimension k and no linear subspace of higher dimension.

For example, $\mathbb{R}_+^2 \cup \mathbb{R}_-^2$ (and more generally, $\mathbb{R}_+^n \cup \mathbb{R}_-^n$ [15]) is a cone of rank 1. A cone C of rank k is called *solid* if its interior is nonempty, and *k-solid* if there is a linear subspace W of dimension k such that $W \setminus \{0\} \subseteq \text{int}(C)$; k-solid cones are useful in the analysis of dynamical systems [11], [12], [14], [33]. Roughly speaking, if a trajectory of the system is confined to an invariant set C that is a k-solid cone then the trajectory can be projected onto a k-dimensional subspace contained in C. If this projection is one-to-one then the trajectory is topologically conjugate to a trajectory of a k-dimensional dynamical system.

It was shown in [38] (see also [23]) that for any $k \in \{1, \ldots, n-1\}$, the set P_-^k is a k-solid cone, and that its complement

$$(P_-^k)^c := \text{clos}(\mathbb{R}^n \setminus P_-^k)$$

is an $(n-k)$-solid cone. This implies, in particular, that there exists a k-dimensional subspace W^k such that $W^k \subseteq P_-^k$, and that there is no $(k+1)$-dimensional subspace contained in P_-^k. For example, let $e^i \in \mathbb{R}^n$ denote the vector with all entries zero, except for entry i that is one. Then, the k-dimensional subspace $\text{span}\{e^1, \ldots, e^k\}$ is contained in P_-^k.

We can now introduce and analyze a new class of DT linear systems.

III. DT k-Positive Linear Systems

In this section, we define a k-positive DT linear system. We then prove two properties of such systems. In Section III-A, we use the spectral properties of nonsingular SSR_k matrices to prove the k-exponential separation property. In Section III-B, we analyze the dynamics of exterior products of solutions.

Definition 1: Consider the DT LTV (1) with every matrix $A(i)$ nonsingular. The system is called *k-positive* if it maps P_-^k to P_-^k. It is called *strongly k-positive* if it maps $P_-^k \setminus \{0\}$ to P_+^k.

Note that (10) implies that a [strongly] positive system is a [strongly] 1-positive system. Note also that since $P_+^k = \text{int}(P_-^k)$, both P_-^k and P_+^k are invariant sets of a strongly k-positive system.

Theorem 1: The system (1) is a [strongly] k-positive system iff $A(i)$ is $[S]SR_k$ for all $i \geq 0$.

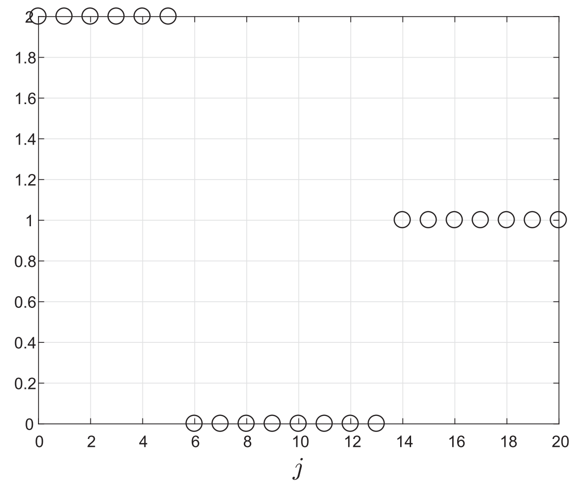

Fig. 1. $s^+(x(j))$ as a function of j for the trajectory in Example 1.

Proof: [4, Th. 1] shows that a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is SSR_k iff for any $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \leq k-1$, we have $s^+(Ax) \leq k-1$. A standard continuity argument [38] shows that A is SR_k iff for any $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \leq k-1$, we have $s^-(Ax) \leq k-1$. ■

For the case $k = 1$ this is a generalization of [strongly] positive linear systems. For example, a system is typically defined as strongly positive if all the entries of $A(k)$ are positive, yet it is strongly 1-positive if all its entries are either all positive or all negative.

Example 1: Consider the system (1) with $n = 4$ and

$$A(i) = \begin{bmatrix} 9 & 2 & -2 & 1 \\ 3 & 10 & 1 & -1 \\ -4 & 1.5 & 12 & 4 \\ 1 & -1 & 2 & 15 \end{bmatrix} \tag{12}$$

for all $i \geq 0$. Note that A is not SSR_1 (as it has both positive and negative entries), nor SSR_2 (as it has both positive and negative minors of order two, e.g., $A(1,2|1,2) = 84$, $A(3,4|1,3) = -20$). All the 16 minors of order three are positive, and $\det(A) \neq 0$, so A is SSR_3 and nonsingular. Fig. 1 shows $s^+(x(j))$ as a function of j for $x(0) = [1\ 1\ {-1}\ 1]^T$. Note that $s^-(x(0)) = 2$. It may be seen that, as expected, $s^+(x(j)) \leq 2$ for all $j \geq 0$. ■

From here on we focus on the time-invariant linear system

$$x(j+1) = Ax(j), \quad x(0) = x_0 \in \mathbb{R}^n \tag{13}$$

where A is nonsingular and SSR_k for some $k \in \{1, \ldots, n-1\}$, leaving the time-varying case and nonlinear systems to a sequel paper. To the best of our knowledge, even for this LTI case our results are new.

A. k-Exponential Separation and Its Implications

Let $C \subseteq \mathbb{R}^n$ be a closed cone that is convex (i.e., $x, y \in C$ implies that $rx + sy \in C$ for all $r, s \geq 0$), and pointed (i.e., $C \cap (-C) = \{0\}$). Then, C induces a (partial) order defined by $a \leq_C b$ if $b - a \in C$. For example, for $C = \mathbb{R}_+^n$ we have $a \leq_C b$ if and only if $b_i \geq a_i$ for all $i \in \{1, \ldots, n\}$. Dynamical systems whose flow preserves such an order are called *monotone* (see, e.g., the excellent monograph [36]).

Since P_-^k and P_+^k are not convex sets, k-positive systems are not monotone systems in the usual sense. However, the fact that P_-^k is a k-solid cone has strong implications for the dynamics of such systems.

The first demonstration of this is a k-exponential separation property of (13). This is closely related to the generalization of Perron's Theorem in [15] (see also [23]) but we give a direct proof based on the spectral properties of a nonsingular SSR_k matrix (see Theorem 2 below). We now review these properties following the presentation in [1].

Fix a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ that is SSR_k for some $k \in \{1,\dots,n-1\}$. Denote the eigenvalues of A by λ_i, $i = 1,\dots,n$, ordered such that

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n| > 0 \tag{14}$$

and let

$$v^1,\ v^2,\dots,v^n \tag{15}$$

denote the corresponding eigenvectors, with complex conjugate eigenvalues appearing in consecutive pairs (we say, with a mild abuse of notation, that $z \in \mathbb{C}^n$ is *complex* if it is not real). We use $\bar{z}$ to denote the complex conjugate of z. We may assume that every v^i is not purely imaginary. Indeed, otherwise we can replace v^i by $\mathrm{Im}(v^i)$ that is a real eigenvector. Also, the fact that A is real means that if v^i is complex then its real and imaginary parts can be chosen as linearly independent.

Define a set of real vectors $u^1, u^2, \dots, u^n \in \mathbb{R}^n$ by going through the v^i's as follows. If v^1 is real then $u^1 := v^1$ and proceed to examine v^2. If v^1 is complex (and whence $v^2 = \bar{v}^1$) then $u^1 := \mathrm{Re}(v^1)$, $u^2 := \mathrm{Im}(v^1)$ and proceed to examine v^3, and so on.

Suppose that for some i, j, the eigenvector v^i is real, and v^j is complex. Then, it is not difficult to show that since A is real and nonsingular, the real vectors $v^i, \mathrm{Re}(v^j)$, and $\mathrm{Im}(v^j)$ are linearly independent.

Note that if $v^i, v^{i+1} \in \mathbb{C}^n$ is a complex conjugate pair and $c \in \mathbb{C} \setminus \{0\}$ is complex then

$$cv^i + \bar{c}v^{i+1} = 2(\mathrm{Re}(c)\,\mathrm{Re}(v^i) - \mathrm{Im}(c)\,\mathrm{Im}(v^i)) \in \mathbb{R}^n \setminus \{0\}$$

so by choosing an appropriate $c \in \mathbb{C} \setminus \{0\}$ we can get any nonzero real linear combination of the real vectors $\mathrm{Re}(v^i)$ and $\mathrm{Im}(v^i)$.

For $p \le q$, we say that a set $c_p,\dots,c_q \in \mathbb{C}$ *matches* the set $v^p,\dots,v^q$ of consecutive eigenvectors (15) if the c_i's are not all zero and for every i if the vector v^i is real then c_i is real, and if v^i, v^{i+1} is a complex conjugate pair then $c_{i+1} = \bar{c}_i$. In particular, this implies that $\sum_{i=p}^q c_i v^i \in \mathbb{R}^n$.

It was shown in [1] that if $A \in \mathbb{R}^{n\times n}$ is nonsingular and SSR_k with signature ϵ_k, then the product $\epsilon_k\lambda_1\lambda_2\dots\lambda_k$ is real and positive

$$|\lambda_k| > |\lambda_{k+1}| \tag{16}$$

and if $c_1,\dots,c_k \in \mathbb{C}$ $[c_{k+1},\dots,c_n \in \mathbb{C}]$ match the eigenvectors $v^1,\dots,v^k$ $[v^{k+1},\dots,v^n]$ of A, then

$$s^+\left(\sum_{i=1}^k c_i v^i\right) \le k-1 \tag{17}$$

$$s^-\left(\sum_{i=k+1}^n c_i v^i\right) \ge k. \tag{18}$$

Furthermore, let $\{u^1,\dots,u^n\}$ be the set of real vectors constructed from $\{v^1,\dots,v^n\}$ as described above. Then, $u^1,\dots,u^k$ are linearly independent. In particular, if $v^1,\dots,v^k$ are real then they are linearly independent.

Example 2: Let

$$A := \begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix}. \tag{19}$$

It is straightforward to verify that this matrix is nonsingular, and SSR_3 with $\epsilon_3 = 1$. Its eigenvalues are[2]

$$\lambda_1 = 3 + s_1,\ \lambda_2 = 3 + \mathrm{i}s_2,\ \lambda_3 = 3 - \mathrm{i}s_2,\ \lambda_4 = 3 - s_1$$

where $\mathrm{i}^2 = -1$, $s_1 := \sqrt{1+4\sqrt{3}} \approx 2.8157$, and $s_2 := \sqrt{-1+4\sqrt{3}} \approx 2.4348$. Note that $\lambda_1\lambda_2\lambda_3$ is real and positive, and that $|\lambda_3| > |\lambda_4|$. The matrix of corresponding eigenvectors is

$$V := [\,v^1\ v^2\ v^3\ v^4\,] = \begin{bmatrix} \frac{s_1-1}{2} & \frac{\mathrm{i}s_2-1}{2} & \frac{-(\mathrm{i}s_2+1)}{2} & \frac{-(s_1+1)}{2} \\ \frac{s_1^2-1}{12} & \frac{-(1+s_2^2)}{12} & \frac{-(1+s_2^2)}{12} & \frac{s_1^2-1}{12} \\ \frac{2}{s_1-1} & \frac{-2(1+\mathrm{i}s_2)}{1+s_2^2} & \frac{2(-1+\mathrm{i}s_2)}{1+s_2^2} & \frac{-2}{s_1+1} \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and thus

$$U := [\,u^1\ u^2\ u^3\ u^4\,] = [\,v^1\ \mathrm{Re}(v^2)\ \mathrm{Im}(v^2)\ v^4\,] = \begin{bmatrix} \frac{s_1-1}{2} & \frac{-1}{2} & \frac{s_2}{2} & \frac{-(s_1+1)}{2} \\ \frac{s_1^2-1}{12} & \frac{-(1+s_2^2)}{12} & 0 & \frac{s_1^2-1}{12} \\ \frac{2}{s_1-1} & \frac{-2}{1+s_2^2} & \frac{-2s_2}{1+s_2^2} & \frac{-2}{s_1+1} \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Note that $s^-(u^i) = s^+(u^i) = i-1$, $i = 1, 2, 4$, and

$$1 = s^-(u^3) < s^+(u^3) = 2.$$

■

We now state the main result in this section. Let $||\cdot|| : \mathbb{R}^n \to \mathbb{R}_+$ denote some vector norm.

Theorem 2: Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and SSR_k for some $k \in \{1,\dots,n-1\}$. Let $u^1,\dots,u^n$ be the real vectors constructed from the eigenvectors of A as described above. Then, there exist subspaces $E := \mathrm{span}\{u^1,\dots,u^k\}$ and E^c such that the following properties hold:

(i) $\dim(E) = k$ and $\dim(E^c) = n-k$;
(ii) both E and E^c are invariant under A;
(iii) $E \subseteq \mathrm{int}(P_-^k) \cup \{0\}$, and $E^c \cap P_-^k = \{0\}$.
(iv) There exist $a > 0$ and $b \in (0,1)$ such that for any $x(0) \in E$, $\tilde{x}(0) \in E^c$, with $||x(0)|| = ||\tilde{x}(0)|| = 1$, the corresponding solutions of (13) satisfy

$$||\tilde{x}(j)|| \le ab^j||x(j)||. \tag{20}$$

(v) For any $x(0)$ satisfying

$$x(0) = f + g, \text{ where } f \in E \setminus \{0\} \text{ and } g \in E^c \tag{21}$$

there exists a $q = q(x(0)) \ge 0$ such that the corresponding solution of (13) satisfies

$$s^+(x(j)) \le k-1 \text{ for all } j \ge q.$$

Remark 1: Condition (v) does not necessarily mean that $x(0)$ is an element of E, as it may also include some nonzero combination of the vectors $u^{k+1},\dots,u^n$ that are not in E. Assertion (v) states that for almost any initial condition, the corresponding solution of the

[2] All numerical values in this article are subject to four-digits accuracy.

dynamical system converges to P_+^k in finite time. Thus, P_+^k is an almost globally attractive invariant set of the dynamics. Invariant sets play an important role in systems and control theory (see, e.g., [6]).

Proof of Theorem 2: We consider without loss of generality the generic case, where $u^1, \dots, u^n$ are linearly independent. Then, $E^c = \text{span}\{u^{k+1}, \dots, u^n\}$. The proofs of the properties of E^c are then very similar to the proofs for E, and thus we present here only the proofs for E.

We begin by noting that the eigenvalues of A are ordered as

$$|\lambda_1| \geq \dots \geq |\lambda_k| > |\lambda_{k+1}| \geq \dots \geq |\lambda_n| > 0. \tag{22}$$

Assertion (i) follows immediately from the fact that $u^1, \dots, u^n$ are linearly independent.

Pick $z \in E \setminus \{0\}$. Since $\prod_{\ell=1}^k \lambda_\ell$ is real, either λ_{k-1}, λ_k are both real, or they are a complex conjugate pair. Combining this with the definition of E implies that $z = \sum_{i=1}^k c_i v^i$, for some $c_1, \dots, c_k$ that match $v^1, \dots, v^k$. Hence, $Az = \sum_{i=1}^k c_i \lambda_i v^i$. Clearly, $\{c_1\lambda_1, \dots, c_k\lambda_k\}$ also match $\{v^1, \dots, v^k\}$, so E is invariant under A.

It follows from (17) and the construction of the u^i's that $s^+(z) \leq k-1$ for any $z \in E \setminus \{0\}$, that is, $E \setminus \{0\} \subseteq P_+^k$. Since $P_+^k = \text{int}(P_-^k)$, we conclude that $E \subseteq \text{int}(P_-^k) \cup \{0\}$.

To prove (iv), pick $x(0) \in E \setminus \{0\}$ and $\tilde{x}(0) \in E^c \setminus \{0\}$. Then, $x(0) = \sum_{i=1}^k c_i v^i$ and $\tilde{x}(0) = \sum_{i=k+1}^n \tilde{c}_i v^i$, where $c_1, \dots, c_k \in \mathbb{C}$ $[\tilde{c}_{k+1}, \dots, \tilde{c}_n \in \mathbb{C}]$ match $v^1, \dots, v^k$ $[v^{k+1}, \dots, v^n]$. Using (22), a straightforward argument shows that there exists $m > 0$ such that

$$\begin{aligned} ||x(j)|| &= ||A^j x(0)|| \\ &\geq m|\lambda_k|^j ||x(0)||. \end{aligned}$$

Similarly, there exists $M > 0$ such that $||\tilde{x}(j)|| \leq M|\lambda_{k+1}|^j ||\tilde{x}(0)||$. Thus

$$\frac{||\tilde{x}(j)||}{||x(j)||} \leq \frac{M}{m} \left| \frac{\lambda_{k+1}}{\lambda_k} \right|^j \frac{||\tilde{x}(0)||}{||x(0)||}$$

and combining this with (16) proves (20).

To prove (v), pick $x(0)$ such that (21) is satisfied. Then, $x(0) = \sum_{i=1}^n c_i v^i$, where $c_1, \dots, c_n \in \mathbb{C}$ match $v^1, \dots, v^n$, and $\sum_{i=1}^k c_i v^i \neq 0$. Thus,

$$\frac{x(j)}{||\sum_{i=1}^k c_i \lambda_i^j v^i||} = \frac{\sum_{i=1}^k c_i \lambda_i^j v^i}{||\sum_{i=1}^k c_i \lambda_i^j v^i||} + \frac{\sum_{i=k+1}^n c_i \lambda_i^j v^i}{||\sum_{i=1}^k c_i \lambda_i^j v^i||}.$$

The first term on the right-hand side of this equation is a unit vector in E, and the second term goes to zero as $j \to \infty$. Thus, there exists $r \geq 0$ such that $x(r) \in P_-^k$. Then, $x(r+1) \in P_+^k$, and the invariance of P_+^k implies that $x(j) \in P_+^k$ for all $j \geq r+1$. ■

The next example demonstrates a simple application of Theorem 2.

Example 3: Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}.$$

We assume that $\det(A) = a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} \neq 0$, so A is nonsingular. It is straightforward to verify that A is SSR_2 iff either a_{31} is negative and all the other a_{ij}'s are positive or if a_{31} is positive and all the other a_{ij}'s are negative. For concreteness, we assume the first case. Note that since $a_{31}a_{11} < 0$ the matrix is not SR_1. The dynamics $x(k+1) = Ax(k)$ represents a cyclic linear system, where the dynamics of each state-variable x_i, $i = 1, 2$, depends on the state of x_i, x_{i+1} in a cooperative manner, and there is a negative feedback from $x_1(k)$ to $x_3(k+1)$. CT cyclic systems have found many applications in various fields (see, e.g. [17]). Theorem 2 implies that for almost every initial condition $x(0)$ the solution $x(k)$ converges to P_+^2, that is

$$s^+(x(k)) \leq 1 \text{ for all } k \text{ sufficiently large.} \tag{23}$$

Note that if we multiply A by any $c \in \mathbb{R} \setminus \{0\}$ then the required sign pattern of A still holds, and thus the same conclusion holds. This implies that (23) is independent of stability. In other words, (23) cannot be used to determine if solutions go to the origin, to a limit cycle, or to infinity, yet it implies that the solutions must go there "through" P_+^2. ■

B. Dynamics of Exterior Products

The *exterior product* (also called wedge product) of vectors is an algebraic construction that can be used to study geometric properties: areas, volumes, and their higher-dimensional analogues [18]. Pick $Z \in \mathbb{R}^{n \times k}$, with $k \leq n$. Denote the columns of Z by $z^1, \dots, z^k \in \mathbb{R}^n$. Then, its kth multiplicative compound $Z^{(k)} \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{k}}$ is the exterior product $z^1 \wedge \dots \wedge z^k$, represented as a column vector [27]. For example, for $z^1 = [\,r_1\ r_2\ r_3\,]^T$ and $z^2 = [\,w_1\ w_2\ w_3\,]^T$, we have

$$\begin{aligned} Z^{(2)} &= \begin{bmatrix} r_1 & w_1 \\ r_2 & w_2 \\ r_3 & w_3 \end{bmatrix}^{(2)} \\ &= [\,r_1w_2 - r_2w_1\ \ r_1w_3 - r_3w_1\ \ r_2w_3 - r_3w_2\,]^T. \end{aligned}$$

Consider the dynamics (13), where $A \in \mathbb{R}^{n \times n}$ is SSR_k, and pick k initial conditions $w^1, \dots, w^k \in \mathbb{R}^n$. Let

$$X(j) := \left[\, x(j, w^1) \ \dots \ x(j, w^k) \,\right] \in \mathbb{R}^{n \times k}. \tag{24}$$

Then, $X(j+1) = AX(j)$. Taking the kth multiplicative compound on both sides of this equation and using (7) yields

$$\eta(j+1) = A^{(k)} \eta(j) \tag{25}$$

where

$$\eta(j) := x(j, w^1) \wedge \dots \wedge x(j, w^k). \tag{26}$$

The magnitude of this wedge product is the volume of the k-dimensional parallelotope whose edges are the given vectors.

Example 4: Suppose that $n = 3$, $A := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, $k = 2$, $w^1 = e^p$, and $w^2 = e^q$ for some $p, q \in \{1, 2, 3\}$. Then

$$\begin{aligned} \eta(j) &= x(j, e^p) \wedge x(j, e^q) \\ &= (\lambda_p^j e^p) \wedge (\lambda_q^j e^q) \\ &= \lambda_p^j \lambda_q^j (e^p \wedge e^q) \\ &= (\lambda_p \lambda_q)^j \eta(0). \end{aligned}$$

This implies that under the dynamics (13) the unsigned area of the parallelogram having e^p and e^q as two of its sides scales as $(\lambda_p\lambda_q)^j$.

On the other hand, $A^{(2)} = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix}$.

If A is SSR_k then either every entry of $B := A^{(k)}$ is positive or every entry is negative. We assume that $B > 0$. By the Perron theorem, the spectral radius of B, denoted $\rho(B)$, is a positive eigenvalue and there exist positive vectors v^B, w^B, such that $Bv^B = \rho(B)v^B$ and $B^T w^B =$

$\rho(B)w^B$. By normalization, we may assume that $(v^B)^T w^B = 1$. Then, furthermore

$$\lim_{j\to\infty} \left(\frac{B}{\rho(B)}\right)^j = v^B (w^B)^T \tag{27}$$

(see, e.g., [19, Ch. 8]). This yields the following result.

Theorem 3: Suppose that A is SSR_k and that $B := A^{(k)} > 0$. Pick k initial conditions $w^1, \dots, w^k \in \mathbb{R}^n$, and define $X(j)$ and $\eta(j)$ as in (24) and (26). Then

$$\lim_{j\to\infty} \frac{\eta(j)}{(\rho(B))^j} = (w^B)^T \eta(0) v^B. \tag{28}$$

Proof: By (25), $\eta(j) = B^j \eta(0)$, i.e., $\frac{\eta(j)}{(\rho(B))^j} = (\frac{B}{\rho(B)})^j \eta(0)$. Taking $j \to \infty$ and using (27) completes the proof. ■

Remark 2: Suppose that the spectral radius $\rho(A)$ of A satisfies $\rho(A) < 1$. Then, $\lim_{j\to\infty} A^j x = 0$ for all $x \in \mathbb{R}^n$ and thus

$$\eta(j) = (A^j w^1) \wedge \cdots \wedge (A^j w^k)$$

satisfies $\lim_{j\to\infty} \eta(j) = 0$. Since every eigenvalue of $A^{(k)}$ is the product of k eigenvalues of A, $\rho(B) < 1$ so (28) also shows that $\eta(j)$ goes to zero as $j \to \infty$.

Example 5: Consider the case $n = 3, k = 2$,

$$A := \begin{bmatrix} 0.79 & 0.2 & 0.01 \\ 0.1 & 0.8 & 0.1 \\ 0.01 & 0.1 & 0.89 \end{bmatrix},$$

$w^1 = e^1$, and $w^2 = e^2$. In other words, we consider the evolution of the unsigned area of the parallelogram with e^1 and e^2 as two of its sides. A calculation yields

$$B := A^{(2)} = \begin{bmatrix} 0.612 & 0.078 & 0.012 \\ 0.077 & 0.703 & 0.177 \\ 0.002 & 0.088 & 0.702 \end{bmatrix}$$

(so A is SSR_2), $\rho(B) = 0.8430$

$$v^B = [0.2991\ 0.8075\ 0.5084]^T$$

and

$$w^B = [0.2203\ 0.6394\ 0.8217]^T$$

(note that $(w^B)^T v^B = 1$). We compute $\eta(15)$ in two different ways. First

$$\begin{aligned} \eta(15) &= (A^{15} e^1) \wedge (A^{15} e^2) \\ &= [0.2397\ 0.2190\ 0.1858]^T \\ &\quad \wedge [0.4228\ 0.4103\ 0.3859]^T \\ &= 0.0057 e^1 + 0.0139 e^2 + 0.0083 e^3. \end{aligned} \tag{29}$$

Second, it follows from (28) that

$$\begin{aligned} \eta(15) &\approx (\rho(B))^{15} (w^B)^T \eta(0) v^B \\ &= (\rho(B))^{15} w_1^B v^B \\ &= \begin{bmatrix} 0.0051 & 0.0137 & 0.0086 \end{bmatrix}^T \end{aligned}$$

and this is indeed an approximation of (29). ■

Recall that if $A \in \mathbb{R}^{n\times n}$ is Schur and $A \geq 0$, then (13) admits a *diagonal* Lyapunov function, that is, there exists a diagonal and positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A^T P A - P$ is negative definite [5]. Diagonal Lyapunov functions play an important role in stability analysis of nonlinear systems and in passivity theory (see, e.g., [21] and [37]). Now assume that A is SSR_k, with $\epsilon_k = 1$, and Schur. Then, there exists a diagonal and positive-definite matrix $D \in \mathbb{R}^{r\times r}$, with $r := \binom{n}{k}$, such that $(A^{(k)})^T D A^{(k)} - D$ is negative definite. In other words, the dynamical system (25) admits a diagonal Lyapunov function. This suggests that in this case many of the powerful ideas based on using diagonal Lyapunov functions can be extended to study the evolution of k-dimensional exterior products.

IV. Discussion

Positive systems and their nonlinear counterpart of monotone systems form a class of dynamical systems of fundamental importance in systems biology, neuroscience, and biochemical networks, and has recently also found important applications in control engineering for large-scale systems [32].

We introduced a new class of DT linear systems that map the set of vectors with up to $k-1$ sign variations to itself. For $k = 1$ this reduces to the important notion of DT positive linear systems.

An interesting research direction is to study DT nonlinear systems whose variational equation is a k-positive linear system. Since the variational equation (3) includes the integral of a matrix, this raises the following question: when is the integral of a matrix SSR_k?

Theorem 3 describes the convergence to a ray for the exterior product. We believe that this can generalized to the DT time-varying linear system (1), with the matrices $A(i)$ taken from a compact set, using the Birkhof–Hopf theory [7].

Another interesting research direction may be the extension of k-positive systems to DT control systems as was done for CT monotone systems in [3]. Finally, our results highlight the importance of an efficient algorithm for determining if a given matrix is SSR_k for some k. This issue is currently under study [2].

Appendix
Proof of Fact 1

Let v^i, $i = 1, 2, \dots$, be a set of vectors such that $v^i \in P_-^k$ for all i and $v := \lim_{i\to\infty} v^i$ exists, Applying (8) yields

$$s^-(v) \leq \liminf_{i\to\infty} s^-(v^i) \leq k - 1$$

so $v \in P_-^k$. We conclude that P_-^k is closed.

To prove (9) pick $z \in P_+^k$. Then, $s^-(z) \leq s^+(z) \leq k-1$, so $z \in P_-^k$. This shows that $P_+^k \subseteq P_-^k$. Thus, it is enough to show that the boundary of P_-^k, denoted ∂P_-^k, satisfies

$$\partial P_-^k = P_-^k \setminus P_+^k. \tag{30}$$

Pick $x \in \partial P_-^k$. Then, $s^-(x) \leq k-1$ and for any $\varepsilon > 0$ there exists $y \in \mathbb{R}^n$ such that $||x - y|| \leq \varepsilon$ and $s^-(y) > k-1$. This implies that we can find a set of vectors x^i, $i = 1, 2, \dots$, such that $\lim_{i\to\infty} x^i = x$ and $s^-(x^i) > k-1$ for all i. Applying (8) yields

$$k - 1 < \liminf_{i\to\infty} s^-(x^i) \leq s^+(x).$$

We conclude that $x \in P_-^k$ and $x \notin P_+^k$, so

$$\partial P_-^k \subseteq P_-^k \setminus P_+^k.$$

Now pick $x \in P_-^k \setminus P_+^k$. Thus, $s^-(x) \leq k-1$ and $s^+(x) > k-1$. This implies that there exists a nonempty set of indexes E such that for every $i \in E$ we have $x_i = 0$ and there exists $r_i \in \{-1, 1\}$ such that the vector y defined by

$$y_k := \begin{cases} r_k, & k \in E \\ x_k, & k \notin E \end{cases}$$

satisfies $s^-(y) > k-1$. Seeking a contradiction, assume that $x \in \operatorname{int}(P_-^k)$. Then, for any sufficiently large $c > 0$ the vector z defined by

$$z_k := \begin{cases} r_k/c, & k \in E \\ x_k, & k \notin E \end{cases}$$

satisfies $z \in P_-^k$. But by $s^-(z) = s^-(y) > k-1$ we obtain a contradiction which shows that $x \in \partial P_-^k$. Since

$$P_-^k \setminus P_+^k \subseteq \partial P_-^k$$

the proof of Fact 1 is completed. ■

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