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RECOGNITION OF MATRICES WHICH ARE SIGN–REGULAR OF A GIVEN ORDER AND A GENERALIZATION OF OSCILLATORY MATRICES

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Abstract. In this paper, rectangular matrices whose minors of a given order have the same strict sign are considered and sufficient conditions for their recognition are presented. The results are extended to matrices whose minors of a given order have the same sign or are allowed to vanish. A matrix A is called oscillatory if all its minors are nonnegative and there exists a positive integer k such that A^k has all its minors positive. As a generalization, a new type of matrices, called oscillatory of a specific order, is introduced and some of their properties are investigated.

1. Introduction

A matrix is called *sign-regular of order* k (denoted by SR_k) if all its minors of order k are non-negative or all are non-positive. It is called *strictly sign-regular of order* k (denoted by SSR_k) if it is sign-regular of order k, and all the minors of order k are non-zero. In other words, all minors of order k are positive or all are negative.[1] We use $\varepsilon_k \in \{-1,1\}$ to denote the common sign of minors of order k. SSR_k matrices have applications in continuous / discrete-time k-positive systems which have been recently defined and analyzed in [2, 27]. In passing, we note that our results are part of a growing body of research on the applications of sign-regularity to the asymptotic analysis of dynamical systems, e.g., [1, 3, 17, 19, 24, 26]. Former applications appeared, e.g., in computer aided geometric design [21] and computer vision [18, Section 3.3].
After the first consideration of SR_k matrices in [16], these matrices have been subject of only a few studies, see, [20], where an elegant criterion for an $n\times k$ matrix, with $k<n$, to be SSR_k is provided, see Theorem 4 below. In [12], the linear programing problem in which all minors of maximal order of the coefficient matrix have the same sign is studied. The spectral properties of nonsingular matrices which are SSR_k are studied in [1]. Also, the results therein are extended to spectral properties of matrices that are SSR_k for several values of k, for example for all odd k. In the papers referred to so far with the exception of [20], no practical criterion for a matrix to be $[S]SR_k$ for some k is given. In our paper we present such a sufficient condition and compare it

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[1]We note that the terminology in this field is not uniform and some authors refer to such matrices as sign-consistent of order k.

with the one given in [20].
A matrix $A \in \mathbb{R}^{n\times m}$ is termed (*strictly*) *sign-regular* (SSR, respectively, SR) if it is $(S)SR_k$ for all $k = 1,\ldots,\min\{n,m\}$. The most important examples of SR $[SSR]$ matrices are the totally nonnegative (*TN*) [totally positive (*TP*)] matrices, that is, matrices with all minors nonnegative [positive]. Such matrices have applications in numerous fields including approximation theory, combinatorics, probability theory, computer aided geometric design, differential and integral equations, and others [9, 11, 16, 22]. If the sign condition applies only to the k-minors, we call the matrices *totally nonnegative of order* k (TN_k) [*totally positive of order* k (TP_k)], i.e., we consider $[S]SR_k$ matrices with $\varepsilon_k = 1$. [2] TN_k matrices appear in the study of shape preserving properties of curves [6].
If $A \in \mathbb{R}^{n\times n}$ is TN, then A is called *oscillatory* if some power A^k of A is TP. The groundwork for the theory of these matrices was laid by Gantmacher and Krein [11]. Specifically, they established a basic and simple criterion for any nonsingular TN matrix to be oscillatory. Furthermore, they showed that if $A \in \mathbb{R}^{n\times n}$ is oscillatory, then A^{n-1} must be TP. The *exponent* of an oscillatory matrix A, denoted by $\exp(A)$, is the least positive integer κ such that A^κ is TP. Oscillatory matrices whose exponent is equal to $n-1$ are completely characterized in [10] by using elementary bidiagonal factorization and planar networks. In [28], this approach is used to derive an explicit expression for the exponent of several classes of oscillatory matrices and an upper bound on the exponent for some classes. Bidiagonalization of general oscillatory matrices is given in [7]. In our paper, we review some of the properties of oscillatory matrices, present a new approach to these matrices through properties of a primitive matrix and the compound matrix, introduce a new class of matrices called oscillatory of order k, and study some of their properties.
The reminder of this paper is organized as follows: In Section 2, we introduce the notation used in our paper and present in Section 3 new sufficient conditions to determine whether a matrix is $[S]SR_k$. Section 4 contains our result on the new class of matrices called oscillatory of order k.

2. Notations

For integers k,n, we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1,2,\ldots,n\}$ (with a mild abuse of notation, we will regard these sequences also as sets). We use the set-theoretic symbols $\cup$ and $\setminus$ to denote somewhat not precisely but intuitively the union and the difference, respectively, of two index sequences, where we consider the resulting sequences as strictly increasing ordered. A measure of the gaps in an index sequence $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k)$ is the *dispersion of* α, defined as $d(\alpha) := \alpha_k - \alpha_1 - k + 1$. If $d(\alpha) = 0$, we call α *contiguous*. For a given matrix $A \in \mathbb{R}^{n\times m}$, $\alpha \in Q_{k,n}, \beta \in Q_{s,m}$, we denote by $A[\alpha,\beta]$ the submatrix of A lying in the rows indexed by α and columns indexed by β; if $k = s$ we denote $A(\alpha,\beta) := \det(A[\alpha,\beta])$. We suppress the brackets associated with an index sequence

[2] We note that both abbreviations are often used for a different notion, namely, to denote matrices which are TN_i or TP_i for all $i = 1,\ldots,k$.

if we enumerate its entries explicitly. If we want to refer to the order k of a submatrix or to a minor, we simply say that it is a *k-submatrix* or a *k-minor*, respectively. A submatrix $A[\alpha|\beta]$ or a minor $A(\alpha|\beta)$ is called *row contiguous* if α is contiguous and β is arbitrary, it is called *column contiguous* if α is arbitrary and β contiguous. If it is both row and column contiguous, we simply say that it is *contiguous*.
For $A \in \mathbb{R}^{n\times n}$, the $\binom{n}{k} \times \binom{n}{k}$ matrix $C_k(A)$ denotes the k^{th} multiplicative compound matrix which contains all the minors of order k of A ordered lexicographically. The Sylvester-Franke Theorem provides a relation between the determinant of A and the determinant of its compound as $\det C_k(A) = \det(A)^{\binom{n-1}{k-1}}$, see, e.g., [25]. If all elements of $A \in \mathbb{R}^{n\times n}$ are positive (nonnegative), the matrix A is called *positive* (*nonnegative*). We order the eigenvalues $\lambda_i, i = 1,\ldots,n$, of a matrix $A \in \mathbb{R}^{n\times n}$ as $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. The superscript T denotes transposition. We mean by $\lfloor x \rfloor$ the greatest integer less than or equal to x.

3. Recognition of square matrices that are strictly sign-regular of a given order

In this section, we introduce a sufficient condition for all minors of order k of a matrix to have the same strict sign. First, we review some determinantal equalities which will be used in this section.

A useful tool for exposing the relations between the minors of a matrix is the following result.

LEMMA 1. *Sylvester's Determinant Identity, see, e.g.,* [22, p. 3] *Let $A \in \mathbb{R}^{n\times m}$. Pick $p \in \{1,\ldots,\min\{n,m\}\}$, $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$. For each $i \in \{1,\ldots,n\} \setminus \{\alpha\}$ and $j \in \{1,\ldots,m\} \setminus \{\beta\}$, let*

$$b_{ij} := A(\alpha \cup \{i\}|\beta \cup \{j\}).$$

Then for any $r \leq \min\{n-p, m-p\}$ the minors of order r of $B := (b_{ij}) \in \mathbb{R}^{(n-p)\times(m-p)}$ satisfy

$$B(i_1,\ldots,i_r|j_1,\ldots,j_r) = \left[A(\alpha|\beta)\right]^{r-1} A(\alpha \cup \{i_1,\ldots,i_r\}|\beta \cup \{j_1,\ldots,j_r\}).$$

The submatrix $A[\alpha|\beta]$ is called the *pivot block*. From Sylvester's Determinant Identity we conclude the following determinant identity.

LEMMA 2. *Let $A \in \mathbb{R}^{n\times m}$ and $i = (i_1,\ldots,i_k) \in Q_{k,n}$ with $d(i) > 0$, and $j = (j_1,\ldots,j_k) \in Q_{k,m}$. Suppose that there exist an integer t such that $i_h < t < i_{h+1}$ for some $h \in \{1,\ldots,k-1\}$. Then for any $s \in \{1,\ldots,h\}$, $l \in \{1,\ldots,k\}$, the following determinant identity holds*

$$\begin{aligned} &A(\{i_1,\ldots,\hat{i}_s,\ldots,i_{k-1}\} \cup \{t\}|\{j_1,\ldots,\hat{j}_l,\ldots,j_k\}) A(\{i_1,\ldots,i_k\}|\{j_1,\ldots,j_k\}) = \\ &A(\{i_1,\ldots,\hat{i}_s,\ldots,i_k\}|\{j_1,\ldots,\hat{j}_l,\ldots,j_k\}) A(\{i_1,\ldots,i_{k-1}\} \cup \{t\}|\{j_1,\ldots,j_k\}) + \\ &A(\{i_1,\ldots,i_{k-1}\}|\{j_1,\ldots,\hat{j}_l,\ldots,j_k\}) A(\{i_1,\ldots,\hat{i}_s,\ldots,i_k\} \cup \{t\}|\{j_1,\ldots,j_k\}). \end{aligned} \tag{1}$$

Here the hat notation indicates that the respective index has to be deleted.

Proof. *Let C be the $(k+1)\times(k+1)$ matrix given by*

$$C := \begin{bmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \dots & a_{i_1 j_k} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_h j_1} & a_{i_h j_2} & \dots & a_{i_h j_k} & 0 \\ a_{t j_1} & a_{t j_2} & \dots & a_{t j_k} & 1 \\ a_{i_{h+1} j_1} & a_{i_{h+1} j_2} & \dots & a_{i_{h+1} j_k} & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ a_{i_{k-1} j_1} & a_{i_{k-1} j_2} & \dots & a_{i_{k-1} j_k} & 0 \\ a_{i_k j_1} & a_{i_k j_2} & \dots & a_{i_k j_k} & 0 \end{bmatrix}.$$

Now apply Sylvester's Determinant Identity, where the pivot block of size $(k-1)\times(k-1)$ is given by all but the s^{th} and last rows and all but the l^{th} and last columns.

LEMMA 3. *Cauchy-Binet Formula, see, e.g.,* [9, Theorem 1.1.1]
Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times m}$, and denote $C := AB$. Then for each pair of index sequences $\alpha \in Q_{k,n}$ and $\beta \in Q_{k,m}$, where $1 \le k \le \min\{n,m,p\}$, we have

$$C(\alpha|\beta) = \sum_{\gamma \in Q_{k,p}} A(\alpha|\gamma)B(\gamma|\beta). \tag{2}$$

REMARK 1. It follows from Lemma 3 that the product of a nonsingular TN matrix with a TP matrix is again a TP matrix.

The following theorem describes spectral properties of a nonsingular matrix that is SSR_k for some value k.

LEMMA 4. [1, Theorem 2] *Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and SSR_k for some value k, with $k \in \{1,\dots,n-1\}$. Then the eigenvalues of A have the following properties.*

(i) The product $\lambda_1\lambda_2\dots\lambda_k$ is real, and $\varepsilon_k\lambda_1\lambda_2\dots\lambda_k > 0$.

(ii) The following inequality holds $|\lambda_k| > |\lambda_{k+1}|$.

The next lemma shows that to verify strict sign-regularity it suffices to check the contiguous minors.

LEMMA 5. [11, Chapter V, Corollary, p.261] *Let $A \in \mathbb{R}^{n\times m}$. Then A is SSR with signature $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{n'})$, where $n' := \min\{n,m\}$, if*

$$0 < \varepsilon_k A(\alpha|\beta), \quad \textit{whenever } \alpha \in Q_{k,n}, \beta \in Q_{k,m} \quad \textit{and} \quad d(\alpha) = d(\beta) = 0, k = 1,\dots,n'.$$

If all the row contiguous and column contiguous k-minors have the same strict sign, then the matrix may not be SSR_k as the following example shows.

EXAMPLE 1. Let $A \in \mathbb{R}^{3\times 3}$,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0.125 & 0.5 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

All the row and column contiguous 2-minors are positive but $A(1,3|1,3) = 0$.

In the following theorem we provide a sufficient condition for matrices to be SSR_k.

THEOREM 1. *Let $A \in \mathbb{R}^{n\times m}$ and the following conditions hold.*

(i) $A[1,\ldots,n|1,\ldots m-1]$ is SSR_{k-1}.

(ii) All row contiguous k-minors of A have the same (strict) sign.

Then A is SSR_k.

Proof. Suppose that Conditions (i), (ii) hold. Assume without loss of generality that the minors in Condition (ii) are negative. We will prove by induction on the dispersion of the row index sequence that all k-minors are negative. Clearly, the result hold for any k-minor $A(\alpha|\beta)$ with $d(\alpha) = 0$. Assume that the claim holds for all k-minors with row dispersion less than r. Pick $\alpha = (i_1,\ldots,i_k) \in Q_{k,n}$ with $d(\alpha) = r$ and $\beta = (j_1,\ldots,j_k) \in Q_{k,m}$. As the sequence α is not composed of consecutive integers, we can add to this sequence an integer t between i_h and i_{h+1} for some $h \in \{1,\ldots,k-1\}$. Applying (1) with $\ell = k$ to A^T yields

$$\begin{aligned} & A(\{i_1,\ldots,\hat{i}_s,\ldots,i_{k-1}\} \cup \{t\}|\{j_1,\ldots,j_{k-1}\})A(\{i_1,\ldots,i_k\}|\{j_1,\ldots,j_k\}) \\ = \; & A(\{i_1,\ldots,\hat{i}_s,\ldots,i_k\}|\{j_1,\ldots,j_{k-1}\})A(\{i_1,\ldots,i_{k-1}\} \cup \{t\}|\{j_1,\ldots,j_k\}) \\ + \; & A(\{i_1,\ldots,i_{k-1}\}|\{j_1,\ldots,j_{k-1}\})A(\{i_1,\ldots,\hat{i}_s,\ldots,i_k\} \cup \{t\}|\{j_1,\ldots,j_k\}). \end{aligned} \quad (3)$$

It follows from condition (i) that the three ($k-1$)-minors in (3) all have all the same strict sign. Now the dispersion of $\{i_1,\ldots,i_{k-1}\} \cup \{t\}$ and $\{i_1,\ldots,\hat{i}_s,\ldots,i_k\} \cup \{t\}$ is strictly less than r. Thus by the induction hypothesis, these minors are also negative. This implies that $A(i_1,\ldots,i_k|j_1,\ldots,j_k)$ is negative, which completes the proof.

Since A is SSR_k if and only if A^T is so, we may conclude the following corollary.

COROLLARY 1. *Let $A \in \mathbb{R}^{n\times m}$ such that the following conditions hold.*

(i) $A[1,\ldots n-1|1,\ldots,m]$ is SSR_{k-1}.

(ii) All column contiguous k-minors of A have the same (strict) sign.

Then A is SSR_k.

REMARK 2. The proof of Theorem 1 shows that in the application of (1) we can choose ℓ as any integer in the range $\{1,\ldots,k\}$. Therefore, the statement of this theorem remains in force if we replace $A[1,\ldots,n|1,\ldots,m-1]$ in Condition (i) by any submatrix which is obtained from A by deletion of one of its columns.

The sufficient condition of Theorem 1 requires to check:

- $\binom{n}{k-1}\binom{m-1}{k-1}$ minors of order $k-1$; and
- $(n-k+1)\binom{m}{k}$ row contiguous k-minors.

We extend Theorem 1 to recognize SR_k matrices.

THEOREM 2. *Let $A \in \mathbb{R}^{n\times m}$ and assume that the following conditions hold.*

(i) A is SR_{k-1} and each selection of $k-1$ rows of A is linearly independent.

(ii) For each contiguous $\alpha \in Q_{k,n}$, $A[\alpha|1,\dots,m]$ is SR_k with common sign ε_k and has rank k.

Then A is SR_k.

Proof. *We follow the proof of Theorem 2.6 in* [22]. *For $h \in (0,1)$, the matrix $Q_m := (h^{(i-j)^2})_{i,j=1}^m$ is TP. Define $U := AQ_m$. It follows from the Cauchy-Binet Formula (2) and the rank conditions that U fulfils the conditions of Theorem 1, and we conclude that U is SSR_k. By definition, $\lim_{h\to 0} Q_m = I$ and thus, $U \to A$ as $h \to 0$. Therefore, A is SR_k which completes the proof.*

REMARK 3. In Theorem 2, the rank condition and the condition on selection of $k-1$ rows of A cannot be waived as the following example shows. Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

and $k=2$. Then $\operatorname{rank}(A)=3$, A is SR_1 and Condition (ii) is satisfied only for $A[3,4|1,2,3]$. A is not SR_2 as it has both positive and negative 2-minors (e.g., $A(3,4|1,2) = 1$ and $A(1,4|2,3) = -1$).

By multiplication of U from the left by the matrix Q_n in the proof of Theorem 2, it can be seen that the strictly sign-regular matrices of order k are dense in the class of sign-regular matrices of order k.

THEOREM 3. *Pick a matrix $A \in \mathbb{R}^{n\times m}$ that is SR_k and of rank r, with $k \le r$. For any $\varepsilon > 0$ there exists a matrix $B \in \mathbb{R}^{n\times m}$ that is SSR_k such that $\| A - B \| \le \varepsilon$, where $\| \,.\, \|$ denotes some matrix norm.*

REMARK 4. By its importance in the theory of linear totally nonnegative differential systems, see, e.g., [26], we consider now the tridiagonal case. For a tridiagonal, entry-wise nonnegative matrix that is SR_k with $\varepsilon_k = 1$ we only need to check whether all its contiguous principal minors up to order k are nonnegative: Let A be a tridiagonal matrix, i.e., $a_{ij} = 0$, whenever $|i-j| > 1$. Then it is readily seen [9,

p. 6] that $A(i_1,\dots,i_k|j_1,\dots,j_k)=0$, whenever there exists an $s\in\{1,\dots,k\}$ such that $|i_s-j_s|>1$. Furthermore, if $A[i_1,\dots,i_k|j_1,\dots,j_k]$ is any submatrix of A that satisfies $|i_s-j_s|\le 1, s=1,\dots,k$, then $A(i_1,\dots,i_k|j_1,\dots,j_k)$ is a product of contiguous principal minors and entries from the super- and/ or subdiagonals. Thus, if A is a tridiagonal, entry-wise nonnegative matrix, then all k-minors of A are nonnegative if all its contiguous principal minors up to order k are nonnegative.

We now give another test for recognizing SSR_k matrices. This is based on the following result.

THEOREM 4. [20, Theorem 2.2] *Let $A\in\mathbb{R}^{n\times k}$ with $n>k$. Then the matrix A is SSR_k if and only if the following $(n-k)k+1$ minors have the same strict sign:*

- *$A(s,s+1\dots,s+k-1|1,\dots,k)$ for any $s\in\{1,\dots,n-k+1\}$,*
- *$A(1,2,\dots,k-r,j,j+1,\dots,j+r-1|1,\dots,k)$ for any r such that $1\le r<k$ and for any j such that $k-r+2\le j\le n-r+1$.*

EXAMPLE 2. Consider the case $n=4$ and $k=2$. Then the $(n-k)k+1=5$ minors that must have the same strict sign are

- $A(1,2|1,2), A(2,3|1,2), A(3,4|1,2)$,
- $A(1,3|1,2), A(1,4|1,2)$.

As an immediate consequence of Theorem 4 we obtain the following corollary.

COROLLARY 2. *Let $A\in\mathbb{R}^{n\times m}$. Then A is SSR_k if and only if for all the $n\times k$ submatrixes of A denoted by A' with column indexes $\{d_1,d_2,\dots,d_k\}\in Q_{k,m}$ the following $(n-k)k+1$ minors of A' have the same strict sign*

$$A'(s,s+1,\dots,s+k-1|d_1,d_2,\dots,d_k)\,\textit{for any}\, s\in\{1,\dots,n-k+1\},$$

and

$$A'(1,2,\dots,k-r,j,j+1,\dots,j+r-1|d_1,d_2,\dots,d_k),$$

for any r such that $1\le r<k$ and for any j such that $k-r+2\le j\le n-r+1$.

By Corollary 2, $\binom{m}{k}((n-k)k+1)$ k-minors have to be checked to decide whether an $n\times m$ matrix is SSR_k. To facilitate the comparison with the amount of k-minors required by Theorem 1, we consider now the case $m=n$ and estimate the computational cost for computing a $(k-1)$-minor as $\frac{1}{k}$ the cost of computing a k-minor (by Laplace expansion, neglecting the multiplication by matrix entries). Then the criterion based on Theorem 1 is superior to the one based on Corollary 2 if the following inequality holds

$$\binom{n}{k}(n-k+1)+\frac{\binom{n}{k-1}\binom{n-1}{k-1}}{k}\le\binom{n}{k}((n-k)k+1),$$

which reduces after simplification to

$$\binom{n}{k-1} \leq n(n-k)(k-1). \tag{4}$$

It turns out that for $n \leq 9$ inequality (4) is satisfied, see Table 1 for the comparison for $n \leq 5$. For larger n, each of both criteria can outperform the other one with the tendency that for relatively large n the criterion based on Corollary 2 is superior to the one based on Theorem 1.

Table 1: Comparison of the number of k-minors in an $n \times n$ matrix required by Theorem 1 and Corollary 2

n	k	no. of k-minors required by	
		Theorem 1	Corollary 2
3	2	9	9
4	2	24	30
4	3	14	16
5	2	50	70
5	3	50	70
5	4	20	25

4. Generalization of oscillatory matrices

In this section, we introduce a new type of matrices, called oscillatory of a specific order, which are intermediate between the nonsingular TN_k and the TP_k matrices. First, we review some properties of oscillatory matrices and present an alternative approach to these matrices through properties of a primitive matrix and the compound matrix.

4.1. Properties of oscillatory matrices

PROPOSITION 1. [11, Chapter II]
Let $A \in \mathbb{R}^{n\times n}$ be an oscillatory matrix. Then A^{n-1} is TP, in particular, A is nonsingular.

PROPOSITION 2. [11, p.102]

(i) The product of two $n \times n$ oscillatory matrices is also an oscillatory matrix with exponent less than or equal to $\lfloor \frac{n}{2} \rfloor$.

(ii) The product of m $n \times n$ oscillatory matrices, with $m \geq n-1$, is TP.

A necessary and sufficient condition for a TN matrix to be oscillatory is given by the following lemma. We will call the lemma the Criterion of Gantmacher and Krein.

LEMMA 6. [11, Chapter II, Theorem 10], [22, Theorem 5.2] *For a TN matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ to be an oscillatory matrix, it is necessary and sufficient that the following two conditions hold:*

(i) A is a nonsingular matrix,

(ii) $a_{i,i+1} > 0$ and $a_{i+1,i} > 0$, $i = 1,2,\ldots,n-1$.

It is clear from the Criterion of Gantmacher and Krein that a nonsingular TN matrix is oscillatory if it is irreducible.

4.2. Alternative approach to oscillatory matrices with application to discrete-time linear-time varying systems

First, we recall from [5, 15] several definitions and results that will be used later on.

DEFINITION 1. A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is said to be *primitive* if it is irreducible and has only one eigenvalue of maximum modulus.

The following theorem characterizes the primitivity of a matrix A.

LEMMA 7. *A nonnegative matrix A is primitive if and only if A^m is positive for some $m \geq 1$.*

The least number m for which A^m is positive holds is called the *exponent of primitivty*. Bounds for m can be found in, e.g., [8, 14]. The following lemma provides a sufficient condition for the primitivity of a matrix.

LEMMA 8. *An irreducible nonnegative matrix with at least one positive entry on its main diagonal is primitive.*

Although an irreducible matrix may have a reducible power, all powers of a primitive matrix are primitive.

LEMMA 9. *Let $A \in \mathbb{R}^{n\times n}$ be nonnegative and primitive. Then A^k is nonnegative and primitive for all $k = 1,2,\ldots$.*

PROPOSITION 3. *Let $A \in \mathbb{R}^{n\times n}$. Then A is oscillatory if and only if for all $k = 1,\ldots,n$ the matrix $C_k(A)$ is nonnegative and primitive.*

Proof. *Suppose that A is oscillatory and $k \in \{1,\ldots,n\}$. Then $C_k(A)$ is nonnegative. Let v be a natural number such that A^v is TP. Then $C_k(A)^v = C_k(A^v)$ is positive and by Lemma 7 $C_k(A)$ is primitive To prove the converse implication, suppose that for all $k = 1,\ldots,n$ the matrix $C_k(A)$ is nonnegative and primitive. Since $C_1(A) = A$, A is irreducible. Furthermore, A is TN and nonsingular, hence A is oscillatory.*

As an application of oscillatory matrices in dynamical systems, the notion of an oscillatory discrete-time system was introduced in [17] which is an important generalization of a totally positive discrete-time system, see, e.g., [1].

DEFINITION 2. [17] The discrete-time linear-time varying (LTV) system

$$y(k+1) = A(k)y(k) \tag{5}$$

with $A : \mathbb{N} \cup \{0\} \mapsto \mathbb{R}^{n\times n}$, is called an *oscillatory discrete time system* $(ODTS)$ *of order* p, if $A(k)$ is oscillatory for all $k \in \mathbb{N}\cup\{0\}$, and every product of p matrices of the form

$$A(k_p)\ldots A(k_2)A(k_1),\ 0 \le k_1 < \ldots < k_p,$$

is TP.

For example, if $A(k)$ is TP for all k then (5) is an $ODTS$ of order 1. Also, by Proposition 2 (ii), (5) is always an $ODTS$ of order $n-1$. To characterize an $ODTS$ of order p, $1 < p < n-1$, the following question arises.
Suppose that we have ℓ matrices, each one oscillatory with exponent less than or equal to p, $1 < p < n-1$. Is every product of p matrices out of these ℓ matrices TP? Of course, the product will be an oscillatory matrix with exponent less than or equal to $\left\lfloor \frac{n}{2} \right\rfloor$. Moreover, the following inequality holds if the matrices $A_1,\ldots,A_p$ commute

$$\exp\{A_1\ldots A_p\} = \min\{\exp(A_1),\ldots,\exp(A_p)\}. \tag{6}$$

By Remark 1, if at least one of the p matrices is TP then their product is TP and (6) always holds. A sufficient condition for $C_k(A_1\ldots A_p)$ being a positive matrix, for $k = 1,\ldots,n$, is that there exist at least two matrices A_s,A_t with $s,t \in \{1,\ldots,p\}, s < t$, and $A_s(\beta|1,\ldots,k)$, $A_t(1,\ldots,k|\beta) > 0$ for all $\beta \in Q_{k,n}, k \in \{1,\ldots,n\}$. Because then all the entries in the first column of $C_k(A_s)$ and the first row of $C_k(A_t)$ are positive, and it follows that the matrix $C_k(A_1\ldots A_p)$ is a positive matrix.

By Theorem 1 and Corollary 1, this requires to calculate $\frac{n(n+1)}{2}$ column contiguous minors of A_s and the same amount of column contiguous minors of A_t. A problem related (via the compound matrix) to the above question concerns the primitivity of a finite set of matrices. A set of m nonnegative matrices $\mathscr{M} = \{A_1,A_2,\ldots,A_m\}$ is *primitive* if $A_{i_1}A_{i_2}\ldots A_{i_k}$ is positive for some indices $i_1,i_2,\ldots,i_k$. For more details, see [4, 13, 23]. The length of the shortest of such products is called the *exponent* of $\mathscr{M}$. The difference to our problem is that in the definition of a primitive set repetitions of the matrices are permitted and that the elements of the set $\mathscr{M}$ need not be primitive.

4.3. A generalization of oscillatory matrices

In this section, we introduce a new type of matrices that are oscillatory of a specific order and investigate some of their properties.

DEFINITION 3. Pick $k \in \{1,\ldots,n\}$. The matrix $A \in \mathbb{R}^{n\times n}$ is called *oscillatory of order* k, denoted by OS_k, if it is nonsingular and *totally nonnegative of order* k, and some power of it is *totally positive of order* k. The smallest such power will be called the *oscillatory exponent of order* k.

Clearly, if A is OS_k for all $k \in \{1, \ldots, n\}$, then A is an oscillatory matrix.

EXAMPLE 3. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 0.5 & 1 & 0 \end{bmatrix}.$$

A is nonsingular and TN_2, and

$$A^2 = \begin{bmatrix} -0.5 & -4 & 0 \\ 2.5 & 2 & -3 \\ 1.5 & 1.5 & -1.5 \end{bmatrix}$$

is TP_2, so A is OS_2 and its oscillatory exponent of order 2 is 2.

In Example 3, the oscillatory exponent is identical with the order. However, both can differ by a large amount: If A is OS_n, then the oscillatory exponent is 1. In the other extreme, if A is OS_1 and is tridiagonal, then the oscillatory exponent is at least $n-1$.

THEOREM 5. *For $k \in \{1, \ldots, n\}$, the matrix $A \in \mathbb{R}^{n \times n}$ is OS_k if and only if the matrix $C_k(A)$ is a nonsingular nonnegative primitive matrix.*

Proof. The proof parallels the one of Proposition 3. The equivalence of the nonsigularity of A and $C_k(A)$ is a consequence of the Sylvester-Franke Theorem.

The Criterion of Gantmacher and Krein gives a necessary and sufficient condition for a TN matrix to be an oscillatory matrix. The following theorem provides a sufficient condition for a TN_k matrix to be an OS_k matrix.

THEOREM 6. *Let $A \in \mathbb{R}^{n \times n}$ be a TN_k matrix. Then A is an OS_k matrix if the following conditions hold:*

(i) A is a nonsingular matrix.

(ii) $A(1, \ldots, k|\beta) > 0$, $A(\beta|1, \ldots, k) > 0$ for all $\beta \in Q_{k,n}$.

Proof. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and TN_k. Then $C_k(A)$ is a nonnegative and nonsingular matrix. From Condition (ii), it follows that the entries in the first row and first column of $C_k(A)$ are all positive. Since the graph associated with $C_k(A)$ is strongly connected we conclude that $C_k(A)$ is irreducible, see, [15, Theorem 6.2.24]. The entry in the top left position is positive and by Lemma 8, $C_k(A)$ is primitive. The statement follows now from Theorem 5.

As a consequence of Theorem 6 and the Cauchy-Binet Formula (2), we obtain that the product of two matrices which are OS_k is again OS_k. The matrix A in Example 1 fulfills the conditions of Theorem 6, and we conclude that A is OS_2. On the other hand, the matrix A in Example 3 shows that the condition (ii) in this theorem is not necessary because $A(1,2|1,3) = 0$.

Invertible transformations on matrices that transform one class onto itself can be quite useful. The next theorem gives three transformations that map an OS_k matrix to an OS_k matrix, where the oscillatory exponent remains invariant.

PROPOSITION 4. *Let $A \in \mathbb{R}^{n\times n}$ be an OS_k matrix, for some $k \in \{1,\dots,n\}$, D and E be diagonal matrices of order n with positive entries and the matrix $T = (t_{ij})$, with $t_{ij} = \delta_{i,n-j+1}$ for $i, j = 1,2,\dots n$, be the backward identity matrix of order n. Then each of the following linear transformations maps OS_k onto OS_k:*

(i) $A \mapsto A^T$,

(ii) $A \mapsto TAT$,

(ii) $A \mapsto DAE$.

Proof. By Theorem 5, it is sufficient to show that $C_k(A^T)$, $C_k(TAT)$, and $C_k(DAE)$ are nonsingular nonnegative primitive matrices. Since $C_k(A^T) = (C_k(A))^T$, the statement (i) is obvious. To prove (ii), it is clear that $C_k(TAT)$ is nonnegative and nonsingular. Let v be a natural number such that A^v is TP_k, then $C_k(TAT)^v = C_k(TA^vT)$ is positive, thus by Lemma 7, $C_k(TA^vT)$ is a primitive matrix. To prove (iii), we note that if A is TN_k, then DAE is TN_k, and if A^v is TP_k, then by using the Cauchy-Binet Formula (2) it follows that $(DAE)^v$ is TP_k, too.

In [1], the spectral properties of nonsingular matrices that are strictly sign-regular for some order are investigated, see Lemma 4. In the following theorem we provide spectral and other properties of OS_k matrices.

THEOREM 7. *Let $A \in \mathbb{R}^{n\times n}$ be OS_k , for some $k \in \{1,\dots,n\}$. Then the following statements are true.*

(i) Each natural power of A is also an OS_k matrix.

(ii) If A has the oscillatory exponent of order k equal to v, then for any integer $\tau \geq v$ the matrix A^τ is TP_k.

(iii) The product $\lambda_1\lambda_2\dots\lambda_k$ is positive.

(iv) The inequality $|\lambda_k| > |\lambda_{k+1}|$ holds.

Proof. Suppose that A is an OS_k matrix for some $k \in \{1,\dots,n\}$. Statement (i) follows from Theorem 5 and Lemma 9. Statement (ii) follows from the application of the Cauchy-Binet Formula (2). Suppose that the oscillatory exponent of order k of A is γ. Then we have by (ii) A^γ and $A^{\gamma+1}$ are TP_k. By Lemma 4 the eigenvalues of these matrices satisfy

$$(\lambda_1\lambda_2\dots\lambda_k)^\gamma > 0$$

and

$$(\lambda_1\lambda_2\dots\lambda_k)^{\gamma+1} > 0$$

from which we obtain (iii). Similarly, statement (iv) can be proved.

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